Human Capital Investment and Optimal Portfolio Choice

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February 1, 2010

Abstract

We analyze how an individual should optimally invest in human capital when he also has financial wealth. We treat the individual’s possibilities to take more education as expansion options and apply real option analysis. In addition, we characterize the individual’s optimal consumption strategy and portfolio weights. The individual has a demand for hedging financial risk, labor income risk, and also wage level risk.

Keywords and phrases: Optimal portfolio choice, Investment in human capital, Hedging demand.


*The authors would like to thank three anonymous referees, Chris Adcock (the editor), and Martin Stibili for useful comments.
1 Introduction

In this paper we examine the decisions of a life-cycle investor that can invest in his/her own human capital, as well as financial assets. Two important characteristics of human capital investments are that they are irreversible and that they have uncertain returns.\footnote{They also differ from many irreversible investments in physical capital in that the investor has a monopoly right to undertake the investment, because the property rights to human capital cannot be transferred (Blinder and Weiss (1976)).} Hence, investments opportunities in human capital are typical examples of projects with inherent option values. This is an important aspect of our analysis; the value of an individual’s human wealth at a given point in time has two components:

1. The value of human capital in place.
2. The value of the options to invest in more human capital at later points in time.

Standard models of saving and portfolio choice either ignore the existence of human capital or treat it as exogenous. Likewise, models of human capital accumulation and labor earnings over the life-cycle usually ignore portfolio choice. This is unfortunate, not only because human wealth is the most important asset for most young and many middle-aged individuals, but in particular because taking into account the interactions between human capital investments, labor income, savings, and portfolio choice can yield important insights into how such decisions are made (or should be made). This paper represents a first step in analyzing these interactions when taking into account the irreversibility of human capital investments. We derive the value of the option to invest in more education and the value of the human capital already in place for an individual. The investor’s total wealth is the sum of these two components of human capital and his financial wealth. The non-marketable risk of human capital causes a demand for hedging its risk. To this end we also characterize the individual’s optimal consumption and portfolio strategy. In addition to the mean-variance tangency portfolio (see e.g. Merton (1969)), the individual now also hedges the risk from labor income and the risk from the rental price for human capital.
Although the existing literature does not analyze simultaneous decisions about saving, portfolio choice and human capital investment, taking into account its option value, there are many studies exploring different aspects of the problem. The portfolio choice analyses of Viceira (2001) and, in particular, Bodie, Merton, and Samuelson (1992) are closely related to this paper. In their models, investors can choose their labor supply. By varying labor supply individuals affect their earnings and therefore the value of their human capital. However, neither Viceira (2001) nor Bodie et al. (1992) analyze human capital investments *per se* and therefore do not take into account the option values inherent in the investor’s wealth. Another related paper is Saks and Shore (2005) on risk and career choice. While their focus is on the interaction between the *type* of education people choose and their portfolio choice, this paper is about the *amount* of education (in a broad sense) and portfolio choice. Judd (2000)\(^2\) and Williams (1978) solve static models of educational investments when the individual also can invest in financial assets. Although both authors recognize the irreversibility of human capital investments, the static framework precludes them from analyzing such investments by the real options approach. Williams (1979) develops an intertemporal model where individuals choose the fraction of time devoted to education and the allocation of financial wealth to different assets. His solution for the optimal portfolio of financial assets has the same structure as in our paper, but the human capital investment decision is quite different as he ignores the option values inherent in the individual’s wealth.

Jacobs (2007) and Hogan and Walker (2007) explicitly includes real options in human capital models. Jacobs shows that real options may increase the required rate of return on human capital investments, while Hogan and Walker conclude that education attainment will be an increasing function of risk associated with education. These radically different conclusion notwithstanding, there is no portfolio decision in either of these models, whereas we focus on the interaction between human capital investments and the allocation of financial wealth.

\(^2\)Note that Judd’s model incorporates moral hazard, while we treat the non-marketablety of human capital as exogenous.
An important contribution to this literature is the careful empirical study of the risk properties of human capital by Palacios-Huerta (2003). He examines the properties of human capital returns for different population groups, and compare these properties with financial returns. One implication of his findings that is particularly relevant for our paper, is that individuals (especially those with less education) can benefit substantially from using financial assets to hedge labor income risk. In this paper, we derive theoretically the structure of such hedging demand, taking into account the option element of human capital.

Finally, our paper is related to the general theory of portfolio choice in the presence of background risk. The background risk in this paper is rental price of human capital (i.e. wage rate) which governs the value of human capital already invested in and the option value of further investment. Labour income is arguably the most important source of background risk. Several studies, including Heaton and Lucas (2000b), Benzoni, Collin-Dufresne, and Goldstein (2007) and Baptista (2008), have examined whether and how labour income affects portfolio choice. In one respect, these papers are more general than ours because they allow for incomplete markets. We will assume that markets are complete, implying that background risk can be priced and capitalized into wealth. On the other hand, the general background risk literature does not take into account that labor income is a function of endogenous, irreversible investments in human capital. This feature of labour income is at the center-stage of our analysis.

The paper is organized as follows: In section 2 we lay out a life-cycle model of human capital investments, savings, and portfolio choice. Assuming that risks to human capital investments are spanned by the traded assets of the economy, we demonstrate in section 3 how an individual’s human wealth should be valued. We also analyze the implied profiles for earnings over the life-cycle. Given these profiles, we proceed to examine the

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3 This literature was initiated by Pratt and Zeckhauser (1987) and Kimball (1993), who provide conditions on utility functions under which the presence of background risk makes people less willing to bear other risks (e.g., portfolio risk). Gollier and Pratt (1996) further examine the relation between utility functions and background risk.

4 Income generated from privately held businesses is another important source of background risk, see e.g. Heaton and Lucas (2000a).
individual’s optimal savings and investment policies in section 4. Section 5 concludes the paper.

2 A life-cycle model

In this section we develop a life-cycle model that will be used to explore the interactions between accumulation of human capital (i.e., education) and portfolio choice.

Preferences and financial wealth We study an individual that is assumed to live forever. The individual derives utility from consumption only, and we treat labor supply as fixed and exogenous. The individual’s objective is to maximize

\[ U_0 = E_0 \left[ \int_0^\infty e^{-\delta t} u(C(t)) dt \right], \tag{1} \]

where \( E_t \) is the conditional expectations operator, \( \delta \) is the rate of time preference, \( C(t) \) is consumption at time \( t \), and \( u \) is an instantaneous utility function with standard properties.

At any time \( t \) the individual can invest in one riskless and one risky financial asset. The riskless asset has an instantaneous real return \( r \), while the price process for the risky financial asset is given by

\[ \frac{dP(t)}{P(t)} = \mu dt + \sigma dz(t). \tag{2} \]

Here, \( z(t) \) is a standard Wiener process, the constant \( \mu \) is the instantaneous expected rate of return on the asset, and \( \sigma \) is the instantaneous standard

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5Infinite time horizon is necessary to obtain a closed-form solution to the option pricing problem considered in the next section. The modelling of human capital formation and labor income presented below implies that human capital investment would tend to occur in early periods of life (if at all) despite the assumption of an infinite horizon.

An interesting alternative between the infinite horizon case analyzed here and the fixed time horizon usually studied in the human capital literature could be to incorporate mortality risk a-la Blanchard (1985). This would potentially yield new insights on the human capital and portfolio choice problem. However, it would complicate the analysis considerably and introduce additional modeling choices that are beyond the scope of the present paper.
deviation of the return. Thus, the risky asset has a lognormal price distribution and normally distributed returns. Modelling financial investment opportunities in this manner is standard and dates back to Merton (1969).

Let $F(t)$ denote the individual’s financial wealth at time $t$, while $\alpha(t)$ gives the share of financial wealth invested in the risky asset. Given (2), it can be shown that the evolution of the financial wealth is given by

$$dF(t) = \left[ (\alpha(t)(\mu - r) + r) F(t) - C(t) \right] dt + \alpha(t) F(t) \sigma dz(t) + dy(t), \quad (3)$$

where $dy(t)$ is the flow of disposable labor income (to be defined below) at time $t$.

**Human capital and labor income** The individual has an initial stock of homogenous human capital (skills and knowledge) $H(0) = H_0$. There is a market in which the services of human capital are traded, and $a(t)$ denotes the real rental price for a unit of human capital $H(t)$ at time $t$. This price is taken as given by the individual. We assume that the rental price $a(t)$ follows the geometric Brownian motion

$$\frac{da(t)}{a(t)} = \lambda dt + b \sigma dz(t), \quad (4)$$

where $\lambda$ is a constant drift coefficient and $b$ is a positive constant.

We follow Bodie et al. (1992) and assume complete markets; the rental price of human capital is perfectly correlated with the risky financial asset return. Admittedly, this is a strong assumption, but is necessary to obtain manageable analytic results. Empirically, labor income growth is not highly correlated with returns to capital, but note that the returns to labour have a very high correlation with the capital returns (see Baxter and Jermann (1997)). Moreover, the assumption of perfect correlation between labor income and risky asset returns is consistent with common production functions used in macroeconomic theory, including the Cobb-Douglas function. In any case, the analytical solutions permitted by the complete markets assumption may provide insights into mechanisms that numerical solutions cannot, and this may be of independent interest.

If the complete market assumption is relaxed, the capitalized value of the individual’s human capital will be individual-specific, depending on the
investor’s preferences and initial financial wealth. Svensson and Werner (1993) provide analytic solutions for consumption and portfolio choice with imperfect correlation between outside income and financial asset returns, but only under the special assumption of exponential utility. Moreover, there is no option element in their analysis. To the best of our knowledge, the option value of investing in more human capital studied in this paper cannot be solved analytically with incomplete markets.

We assume that human capital transforms into labor income through the Cobb-Douglas function

\[ \hat{y}(t) = a(t)H(t)^\theta, \quad 0 < \theta \leq 1, \]  

where \( \hat{y}(t) \) is realized labor income at time \( t \). We notice that the marginal labor earnings product from human capital is \( a\theta H^{\theta-1} \).

Next, we will assume that the individual can add to the human capital stock at any time \( t \) at a cost (in terms of the consumption good) \( k(t) = k_0 e^{\rho t} \) per unit added, with \( \rho > 0 \) and \( k_0 \) a constant. The cost of increasing the level of skills and knowledge is rising over time; human capital is only partially expandable.\(^6\) It is this assumption that ensures reasonable life-cycle behavior, in the sense that human capital investment will tend to occur early in life, despite our infinite time horizon setting.\(^7\) We let \( dH(t) = Q(t)dt \) be the flow of acquired human capital at time \( t \). For simplicity, we will ignore depreciation, so that \( Q(t) \) denotes net investment in human capital at time \( t \). Then, by applying Ito’s lemma to (5), we can write the flow of disposable (for consumption and investment in financial assets) labor income as

\[ dy(t) = a(t)H(t)^\theta \left[ \lambda dt + bσdz(t) \right] - \left[ k_0 e^{\rho t} - a(t)\theta H(t)^{\theta-1} \right] Q(t)dt. \]  

The first term on the right hand-side of (6) is the labor income flow delivered by the preexisting level of human capital, while the second term shows the

\(^6\)Dixit and Pindyck (1998) present a model with partial expandability of physical capital.

\(^7\)Our model implies that the marginal cost of human capital investments shifts upward over time, while there is no time trend in the marginal value of human capital. In classical human capital models with finite time horizon (see e.g., Ben-Porath (1967)), the marginal investment cost is constant over time, while the marginal value shifts down due to the finite horizon. We come back to this issue in Section 3 below.
net income from any investment in human capital at time $t$. The term in the last square brackets is *net* marginal investment cost at time $t$.

Later, it will be convenient to define this term as $I(H,a,t)$, with the properties $\partial I/\partial H > 0$, $\partial I/\partial a < 0$, and $\partial I/\partial t < 0$. The higher the rental price and the lower is the human capital stock, the higher is the immediate income flow delivered by the marginal unit, and thus the lower is net investment costs. The net investment cost increases with time since the unit cost $k$ increases exponentially over time.

3 Human wealth and labor income over the life-cycle

The characteristics of human capital investments make them well suited to be analyzed by the real options approach (see e.g., Dixit and Pindyck (1993)). In this section we use this approach to derive the value of the individual’s human wealth, the optimal human capital investment policy, and the implied profile for labor earnings. The next section incorporates this into the individual’s broader savings and portfolio choice problem.

3.1 Valuation of human wealth

At any time $t$, the value of the individual’s human wealth consists of two components: the value of the human capital already in place, and the option value (evaluated at time $t$) of investing in more human capital now or in the future. We will value each component in turn.

**Human capital in place**  In general, the value at time $t$ of the existing human capital is the expected discounted value of the (maximum) future wage income it can deliver. At time $t$, the individual’s stock of human capital is $H(t)$. If the individual makes no new investments in human capital, this stock will be constant over time. However, the rental price will fluctuate, so

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8The word net is important here. As the model is set up, acquiring a marginal unit of human capital gives an asset with a certain value (to be determined later in this section), but it also gives immediate income $a\theta H^{\theta-1}$. The net investment cost of a marginal unit is thus the gross cost $k$ less the instantaneous income provided by the unit.
at time $\tau$, $t \leq \tau$, labor earnings are $y(\tau) = a(\tau)H(t)^{\theta}$. Combining (4) and (5), we have
\[ dy/y = \lambda dt + b\sigma dz. \tag{7} \]

We can then follow Bodie et al. (1992) (see their section 4) to demonstrate that the value of the individual’s human capital in place is given by
\[ V(H; a, t) = \frac{y(t)}{r + b(\mu - r) - \lambda} = \frac{a(t)H(t)^{\theta}}{r + b(\mu - r) - \lambda}, \tag{8} \]
where we assume that the denominator is positive. Note that the marginal value of acquired human capital at time $t$ is
\[ v(H; a, t) \equiv \frac{\partial V}{\partial H} = \frac{\theta a(t)H(t)^{\theta-1}}{r + b(\mu - r) - \lambda}, \tag{9} \]
a concave function in $H$.

**Option value of investing in more human capital** We now analyze the individual’s options to invest in additional human capital. Denote the value of these options by $G(H; a, t)$. Since we have assumed complete markets, we can follow Dixit and Pindyck (1993) and Dixit and Pindyck (1998) and set up a risk free portfolio to determine $G$. Suppose that we hold one unit of the (portfolio of) expansion options and sell short $m$ units of the spanning asset $n$. In the appendix, we demonstrate that this gives the following differential equation for the value of the marginal expansion option $g(H; a, t) \equiv -\partial G/\partial H$: 
\[ \frac{1}{2}(b\sigma)^2a^2\frac{\partial^2 g}{\partial a^2} + [\lambda - b(\mu - r)]a\frac{\partial g}{\partial a} - rg + \frac{\partial g}{\partial t} = 0. \tag{10} \]
This partial differential equation is subject to the four boundary conditions
\[ g(H; 0, t) = 0, \tag{11} \]
\[ g(H; a^*, t) = v(H; a^*, t) - I(H; a^*, t), \tag{12} \]
\[ \frac{\partial g(H; a^*, t)}{\partial a} = \frac{\partial v(H; a^*, t)}{\partial a} - \frac{\partial I(H; a^*, t)}{\partial a}, \tag{13} \]
\[ \lim_{t \to \infty} g(H; a, t) = 0. \tag{14} \]

\[ ^9 \text{An increase in } H \text{ means exercising some of the future expansion options, so } \partial G/\partial H \text{ must be negative.} \]
If $a$ hits zero, it will stay at zero and the opportunity to invest in human capital will be worthless; hence the first boundary condition. Equations (12) and (13) are the value matching and smooth pasting conditions, respectively (see e.g., Dixit and Pindyck (1993)). The former says that at the value $a^*$ where it is optimal to exercise the marginal option, the individual receives a net payoff equal to the present value of labor income it delivers minus the net marginal cost. The smooth pasting condition requires that $g(H; a, t)$ is continuous and smooth at the critical exercise point $a^*$; if not one could do better by exercising at a different point. Finally, boundary condition (14) says that the value of the option to invest in a marginal unit of human capital approaches zero as time passes by. This follows since the cost of exercising the option (the ‘strike price’) is increasing exponentially with time.

We demonstrate in the appendix that the solution to (10) is given by

$$g(H; a, t) = B(H)a(t)^{\beta_1}e^{-qt},$$

(15)

where $B(H)$ and $q$ are parameters to be determined, and $\beta_1 > 1$ is given in the appendix. Given (15), we can use (12) and (13) to solve for the critical exercise value $a^*$

$$a^*(H, t) = \frac{\beta_1}{\beta_1 - 1} \frac{[r + b(\mu - r) - \lambda]k_0e^{rt}}{\theta H(t)^{\theta-1}}.$$  

(16)

The product $[r + b(\mu - r) - \lambda]k_0e^{rt}$ in (16) can be interpreted as the instantaneous flow cost of increasing the human capital stock by a marginal unit at time $t$. Equation (16) illustrates that the value of the current marginal labor earnings product $a\theta H^{\theta-1}$ must be a multiple $\beta_1/(\beta_1 - 1) > 1$ of this flow cost to trigger investment. The rental price of human capital must cover the full cost of investing: the direct flow cost plus the opportunity cost of investing now instead of later. Notice that the critical exercise value $a^*$ increases over time because the direct investment cost $k(t)$ increases deterministically with time. Moreover, the critical exercise value is also increasing in the level of human capital $H(t)$, because of decreasing marginal labor earnings product.

In Figures 1 and 2, we illustrate these effects graphically. The figures are based on the parameter values reported in Table 1. In addition, we let $t = 0$ correspond to an age of 20 years.
Table 1: Parameter values for numerical illustrations.

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<th>λ</th>
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<td>1.5</td>
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Figure 1 shows the critical rental price (wage rate) $a^*$ for agents of different age and with different levels of human capital $H$. We see how $a^*$ increases with age and with the human capital level. Older individuals and individuals with more human capital require higher wage rates to undertake human capital investments. Figure 2 shows $a^*$ for agents of different age and with different values of $ρ$. If the cost of investment is known to increase rapidly (high $ρ$), a young individual (age 20 in Figure 2) would require a lower wage rate to invest, since it knows that costs tomorrow will be high. Older agents with a high $ρ$ will face high investment cost, and thus need a high wage rate $a$ to invest.

By substituting (16) into (13), we obtain the following expression for
Figure 2: Value of $a^*$ for different ages and growth rates of investment cost $\rho$.

$$B(H) = (\beta_1 - 1)^{\beta_1 - 1} \left( \frac{\theta H^{\theta - 1}}{r + b(\mu - r) - \lambda} \right)^{\beta_1} k_0^{1 - \beta_1} e^{[q - \rho(\beta_1 - 1)]t}. \quad (17)$$

Since $B(H)$ does not depend on $t$, we have from (17) that $q = \rho[\beta_1(q) - 1]$. Inserting for $\beta_1(q)$ and solving for $q$ gives

$$q = \rho \left[ -\frac{1}{2} - \frac{\lambda - b(\mu - r) - \rho}{(b\sigma)^2} + \sqrt{\left( \frac{\lambda - b(\mu - r) - \rho}{(b\sigma)^2} + \frac{1}{2} \right)^2 + \frac{2(r + b(\mu - r) - \lambda)}{(b\sigma)^2}} \right] > 0. \quad (18)$$

Now, (15) gives the value of the option to produce a marginal unit of human capital at time $t$. In principle, the individual could produce an infinite amount of human capital, but we recall that the marginal labor earnings product is decreasing in the preexisting level of human capital, and
this effect enters through the ‘constant’ in (17). Using (15), the total value of the expansion options, evaluated at time $t$ is

$$G(H; a, t) = \int_{H(t)}^{\infty} B(H) a_H(t) \beta_1 e^{-qt} dH$$

$$= \left( \frac{\theta a(t)}{r + b(\mu - r) - \lambda} \right)^{\beta_1} \left( \frac{\beta_1 - 1}{k_0} \right)^{\beta_1 - 1} \int_{H(t)}^{\infty} h^{(\theta - 1)\beta_1} dh.$$ 

A sufficient condition for the latter integral to converge is that $\beta_1 > 1/(1 - \theta)$. That is, if the increase in labor income from more human capital decreases sufficiently fast ($\theta$ is sufficiently less than 1), we can be sure that the individual’s expansion options have finite value for all $t$. We will throughout the paper assume that this convergence condition holds. It then follows that the total value of the expansion options is given by

$$G(H; a, t) = \left( \frac{\beta_1 - 1}{k_0} \right)^{\beta_1 - 1} H(t)^{1-(1-\theta)\beta_1} \left( \frac{\theta a(t)}{r + b(\mu - r) - \lambda} \right)^{\beta_1} e^{-qt}. $$

(18)

The value of the expansion options is increasing in the rental price, decreasing in the stock of human capital, and will approach zero as time passes by since the cost of investing in skills and knowledge rises exponentially with time.

In Figure 3, we illustrate how the value of expansion options varies with age and the preexisting level of human capital. The figure is based on the parameter values reported in Table 1, with the addition of $a(0) = 0.1$. We note that the option value decreases rapidly with age for young agents, and that, given our choice of parameters, the expansion options are of little value for older individuals.

### 3.2 Investment in human capital and labor earnings

The option valuation in the previous subsection is helpful in analyzing the optimal human capital investment policy for the individual. The function $a^*(H, t)$ implicitly defines the optimal human capital level at every instant. If, at time $t$, $a$ and $H$ are such that $a(t) > a^*(H, t)$, the individual should invest in human capital until $a^* = a(t)$. Equivalently, we can rearrange (16) in terms of $H^*(a, t)$, and express the optimal level of human capital at time
Figure 3: Value of expansion option for different ages and levels of human capital $H$.

Thus, if $H^* > H(t)$, the individual would add to the human capital stock, whereas no investment would be undertaken if $H^* \leq H(t)$. Summarizing, the optimal human capital investment policy can be written as

$$Q^*(a, t; H) = H^*(a, t) - H(t), \quad \text{if } a^* < a(t),$$

$$= 0 \quad \text{if } a^* \geq a(t). \quad (20)$$

Given $a^* < a(t)$, the amount of human capital investment is increasing in the rental price $a$ and decreasing with time $t$. As in the standard human capital model (e.g., Ben-Porath (1967)), investment in human capital tends to occur early in life. In the standard model, this investment behavior occurs because the marginal value of human capital decreases as the individual approaches his time horizon. We have an infinite horizon, but with
exponentially increasing investment costs over time the individual still has 
an incentive to undertake human capital investments early.

Perhaps more interestingly, our model has the opposite prediction for the 
relationship between the wage rate (i.e., the human capital rental price) and 
human capital investment, compared to the standard model. The standard 
model predicts a negative link between wage changes and human capital 
investments, because a higher wage rate increases the opportunity cost of 
schooling. Our model predicts that the individual would invest when, ceteris 
paribus, the wage rate is high, because expected returns to education are 
higher. To give a simple example, our model predicts that the popularity of 
MBA programs is high when salaries at Wall Street are high, whereas the 
standard model predicts that MBA programs would be relatively unpopular 
in booms because it is expensive not to work. Which of the two predictions 
that is the most realistic is an empirical question.
Figure 5: Human capital level $H$ for an individual following the optimal investment rule $Q^*$, when facing the wage rate path labeled $a$.

To illustrate how human capital investments may evolve over time, Figures 4 and 5 show the human capital level for a given individual that follows the optimal investment rule (other parameter values are as in Table 1). The two figures differ only with respect to the randomly realized rental price (wage rate) path. (Note that unlike in Figures 1-3, Figures 4 and 5 follow a given individual from age 20 to 45.) Figures 4 and 5 show how different wage realizations give rise to dramatically different levels of human capital. In Figure 4, the wage rate is relatively flat, yielding only a moderate increase in human capital for the time span in question. Investments occur three times: when the agent is young (age 20 in the figure) and twice after a period of rapid wage increase (when the agents is in the late twenties). Figure 5 shows a rental price path that is particularly good for the individual; the wage rate increases more than sixfold over the 25-year period. This individual would make big human capital investments in his mid twenties,
and again in the early forties despite that investment costs at that age are high.

The investment function (20) can be compared to the intertemporal human capital model of Williams (1979). In his model, investors choose the fraction of time devoted to human capital investment at every instant, while in this paper they choose directly how much to add to their preexisting human capital stock. Moreover, while in Williams’s model human capital investment is (partly) driven by the current expected return to education, (20) predicts that investment is driven by the current wage rate (rental price) compared to the trigger rate \( a^* \). Option considerations are not a part of human capital investments in Williams (1979). The presence of \( (\beta_1 - 1)/\beta_1 \) shows that the investment rule in (20) takes the option element into account.

Combining (6) with the optimal investment policy in (20) we have that the optimal disposable labor income flow is given by

\[
d y^* = aH^\theta (\lambda dt + b\sigma dz) + j \left[ (H^* - H) \left( a\theta H^{\theta-1} - k_0 e^{\rho t} \right) \right] dt,
\]

where \( j \) is an indicator function with \( j = 1 \) if \( a(t) \geq a^* \), and \( j = 0 \) otherwise.

4 Consumption and portfolio choice

We now turn to the consumption and portfolio decision of the individual. Formally, the problem is to choose paths for \( C(t) \), \( Q(t) \), and \( \alpha(t) \) to maximize the expected utility in (1), subject to the evolution of the state variables. We have already found the optimal solution for \( Q(t) \) (equation (20)). At this stage we can incorporate this solution in the individual’s maximization problem, upon which the problem is reduced to a pure consumption/portfolio choice problem.

The indirect utility function for this problem is defined by

\[
J(F, a, H, t) = \max_{\{\alpha(\tau), C(\tau)\}} E_t \left[ \int_t^\infty e^{-\delta \tau} u(C(\tau)) d\tau \right],
\]

and the maximization is subject to the budget equation

\[
\begin{align*}
dF &= \left[ (\alpha(\mu - r) + r) F - C \right] dt + \alpha F \sigma dz \\
&\quad + aH^\theta [\lambda dt + b\sigma dz] + jQ^*(H; a, t) \left( a\theta H^{\theta-1} - k_0 e^{\rho t} \right) dt,
\end{align*}
\]
equation (4), the (optimal) evolution of human capital, \( dH^* = jQ^* dt \), and
the current values \( F(t) \), \( a(t) \), and \( C(t) \). The Hamilton-Jacobi-Bellman equation
and first order conditions associated with this problem are reported in
the appendix. The first order conditions can be used to determine the optimal
consumption function and the optimal portfolio demand. These are

\[
C^*(F, a, H, t) = u_C^{-1}(J_F e^{\delta t})
\]  

(21)

and

\[
(\alpha F)^*(F, a, H, t) = -\frac{J_F}{J_{FF}} \frac{\mu - r}{\sigma^2} - by(t) - \frac{J_{Fa}}{J_{FF}} ba(t),
\]  

(22)

respectively.

We recognize (21) as the envelope condition. At the optimum an extra unit of consumption is as valuable to the investor as an extra unit of wealth to finance future consumption. From (22) we see that the optimal portfolio can be decomposed into three terms (Svensson and Werner (1993)):

1. The first term on the right hand-side is usual mean-variance tangency portfolio. It is the reciprocal of the coefficient of absolute risk aversion \(-J_F/J_{FF}\) times the expected excess return to the variance of the risky financial asset.

2. The second term can be labeled the “labor income hedge portfolio”. It gives the substitution away from the risky asset needed to (perfectly) hedge the variability of labor income.

3. The third term gives the adjustment necessary to hedge the uncertainty associated with the state variable \(a\). This term can thus be labeled the “human capital rental price hedge portfolio”, where \(J_{Fa}/J_{FF}\) is the ratio of the absolute aversion of rental price risk to the absolute aversion to (financial) wealth risk.

We note that the portfolio demand in (22) has the same structure as in the human capital model of Williams (1979), see his equation (11), except that we can decompose hedging demand into terms involving the labor income flow and the price of labor. The weight attached to the labor income hedge portfolio is always negative due to the assumption of perfect correlation between the wage rate and risky asset returns. This portfolio thus
contribute to lower financial risk taking. The weight associated with the human capital rental price hedge portfolio is ambiguous. However, for the plausible case in which the investor regards a higher wage rate as a substitute for financial wealth so that $J_{Fa} < 0$, this hedging portfolio is also negative (recall that $J_{FF} < 0$).

5 Conclusion

We have analyzed how an individual should optimally invest in human capital, consume, and construct the financial portfolio. The individual’s wealth has been divided into human wealth and financial wealth. The human wealth has further been divided into human capital already in place and value of the option to invest in more education. We show that there exists a wage level $a^*$, i.e., rental price for human capital, that triggers the individual to invest in more education. When taking human capital into account, this influences on the individual’s optimal portfolio weights. The individual now also hedges the labor income and the uncertainty in the wage level.

A Derivation of selected equations

A.1 Equation (10)

The portfolio of expansion options and $m$ units of the risky financial asset costs $G(H; a, t) - mP(t)$ to buy. Suppose that the options go unexercised at time $t$. The portfolio pays no dividend, but by Itô’s lemma, (2) and (4), instantaneous capital gains are

$$dG - mdP = \left( \frac{\partial G}{\partial t} + \lambda a \frac{\partial G}{\partial a} + \frac{1}{2} (b \sigma)^2 a^2 \frac{\partial^2 G}{\partial a^2} - m \mu P \right) dt$$

$$+ \left( b \sigma a \frac{\partial G}{\partial a} - m \sigma P \right) dz.$$

By choosing $m = \frac{ba \partial G}{P \partial a}$ at every instant, the portfolio will be risk free. In the absence of arbitrage we must accordingly have

$$\left( \frac{\partial G}{\partial t} + \lambda a \frac{\partial G}{\partial a} + \frac{1}{2} (b \sigma)^2 a^2 \frac{\partial^2 G}{\partial a^2} - \frac{ba \partial G}{P \partial a} \mu P \right) dt$$

$$= r \left( G - \frac{ba \partial G}{P \partial a} P \right) dt.$$
Finally, we differentiate this expression with respect to $H$, use the definition $g(H; a, t) = -\partial G/\partial H$, and rearrange to obtain (10).

**A.2 Equation (15)**

By substitution, we can readily confirm that the function $g = Ba(t)^\beta e^{-qt}$ satisfies (10), provided that $\beta$ is a root of

\[ \frac{1}{2}(b\sigma)^2 \beta (\beta - 1) + [\lambda - b(\mu - r)] \beta - r - q = 0. \]

The first root $\beta_1$ is given by

\[ \beta_1 = \frac{1}{2} - \frac{\lambda - b(\mu - r)}{(b\sigma)^2} + \sqrt{\left(\frac{\lambda - b(\mu - r)}{(b\sigma)^2} - \frac{1}{2}\right)^2 + \frac{2(r + q)}{(b\sigma)^2}} > 1, \]

while the second is

\[ \beta_2 = \frac{1}{2} - \frac{\lambda - b(\mu - r)}{(b\sigma)^2} - \sqrt{\left(\frac{\lambda - b(\mu - r)}{(b\sigma)^2} - \frac{1}{2}\right)^2 + \frac{2(r + q)}{(b\sigma)^2}} < 0. \]

The general solution to (10) is thus

\[ g(H; a, t) = Ba^{\beta_1} e^{-qt} + \tilde{B}a^{\beta_2} e^{-qt}, \]

but since $\beta_2 < 0$ the first boundary condition implies that $\tilde{B} = 0$, and we are left with (15).

**B The HJB equation and first order conditions**

The Hamilton-Jacobi-Bellman equation for the consumption/portfolio choice problem is

\[
0 = \max_{\{\alpha, C\}} \{u(C)e^{-\delta t} + J_F[(\alpha(\mu - r) + r)F - C + aH^\theta \lambda + jQ^*(a\theta H^{\theta - 1} - k_0 e^{\rho t})]
+ J_t + \frac{1}{2} J_{FF}[(\alpha F \sigma)^2 + 2\alpha FaH^\theta b\sigma^2 + (aH^\theta b\sigma)^2] + J_a a\lambda + \frac{1}{2} J_{aa}(ab\sigma)^2
+ J_H jQ^* + J_{Fa} ab\sigma^2(\alpha F + baH^\theta)\},
\]

where subscripts denote partial derivatives with respect to the designated variables. The resulting first order conditions are

\[ u_C(C)e^{-\delta t} = J_F \]
and

\[ J_F(\mu - r) + J_{FF}\sigma^2(\alpha F + baH^0) + J_{Fa}ab\sigma^2 = 0. \]

Using \( \tilde{y}(t) = a(t)H(t)^0 \), the consumption function (21) and portfolio demand (22) follows immediately.

References


